

Differential Reduction Algorithms for Hypergeometric Functions Applied to Feynman Diagram Calculation

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We describe the application of differential reduction algorithms for Feynman Diagram calculation. We illustrate the procedure in the context of the generalized hypergeometric functions ${}_{p+1}F_p$, and give an example for a type of q -loop bubble diagram.

1 Introduction

A Feynman diagram can be understood mathematically as a linear combination of Horn-type hypergeometric functions of several variables (see Ref. [2]):

$$\sum_j B_j x_1^{\alpha_1} \cdots x_r^{\alpha_r} \Phi(\vec{\gamma}, \vec{\sigma}, \vec{x}) , \quad (1)$$

where a Horn-type hypergeometric function has the structure

$$\Phi(\vec{\gamma}; \vec{\sigma}; \vec{x}) = \sum_{m_1, m_2, \dots, m_r=0}^{\infty} \left(\frac{\prod_{j=1}^K \Gamma(\sum_{a=1}^r \mu_{ja} m_a + \gamma_j)}{\prod_{k=1}^L \Gamma(\sum_{b=1}^r \nu_{kb} m_b + \sigma_k)} \right) x_1^{m_1} \cdots x_r^{m_r} , \quad (2)$$

with the arguments x_j being, in general, rational functions (typically, simple ratios) of kinematic invariants of the original Feynman diagram, and the parameters $\{\gamma_j\}$ and $\{\sigma_k\}$ being linear combinations of the exponents of propagators and the dimension of space-time.^a The γ_j and σ_k are called upper and lower parameters, respectively.

These statements follow from the multiple Mellin-Barnes representation for a dimensionally regularized Feynman diagram (see Ref. [4]), and the assumption that there is a region of variables where every term in the linear combination (1) is convergent. The Horn-type structure permits the hypergeometric functions appearing in (1) to be reduced to a set of

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^aThe presence of a nontrivial numerator in the Feynman diagram does not affect this conclusion [3].

basis functions with parameters differing from the original ones by integer shifts:

$$P_0(\vec{x})\Phi(\vec{\gamma} + \vec{l}; \vec{\sigma} + \vec{s}; \vec{x}) = \sum_{m_1, \dots, m_r=0}^{\sum_j |l_j| + \sum_k |s_k|} P_{r_1, \dots, r_p}(\vec{x}) D_1^{m_1} \dots D_r^{m_r} H(\vec{\gamma}; \vec{\sigma}; \vec{x}), \quad (3)$$

where $D_j^r = \left(\frac{\partial}{\partial x_j}\right)^r$ denotes a partial derivative and $P_{r_1, \dots, r_p}(\vec{x})$ are rational functions [5]. These shifts may be implemented by constructing a set of four differential operators $U_{\gamma_c}^{\pm}, L_{\sigma_c}^{\pm}$ which respectively change γ_c, σ_c by ± 1 : $\gamma_c \rightarrow \gamma_c \pm 1$ or $\sigma_c \rightarrow \sigma_c \pm 1$. These basic operators are called the step-up and step-down operators for the upper and lower parameters. A procedure of applying step-up and step-down operators to reduce the original hypergeometric function to a basis set is called a **differential reduction**. In the case when some of the variables are equal to one another, $x_i = x_j, i \neq j$, or belong to the surface of singularities $Q = \{\vec{x} | P_0(\vec{x}) = 0\}$, it is necessary to define a limiting procedure for (3).

We will illustrate our approach by considering the reduction of a particular generalized hypergeometric function in section 2, and by applying it to a particular class of Feynman diagrams in section 3. Section 4 describes the reduction at a singular surface using the ε expansion and hyperlogarithms.

2 Generalized hypergeometric function of one variable

In this section, we will show how differential reduction may be applied to a generalized hypergeometric function of one variable. Let us recall that the generalized hypergeometric function ${}_pF_{p-1}(a; b; z)$ may be defined in a neighborhood of $z = 0$ by the series

$$F(\vec{a}; \vec{b}; z) \equiv {}_pF_{p-1} \left(\begin{matrix} \vec{a} \\ \vec{b} \end{matrix} \middle| z \right) = \sum_{k=0}^{\infty} \frac{z^k}{k!} \frac{\prod_{i=1}^p (a_i)_k}{\prod_{j=1}^{p-1} (b_j)_k}, \quad (4)$$

where $(a)_k = \Gamma(a+k)/\Gamma(a)$ is called a Pochhammer symbol. The lists $\vec{a} = (a_1, \dots, a_p)$ and $\vec{b} = (b_1, \dots, b_q)$ are called upper and lower parameters of the hypergeometric function, respectively. The hypergeometric function ${}_pF_{p-1}$ satisfies a differential equation

$$L(\vec{a}, \vec{b}) \left({}_pF_{p-1}(\vec{a}; \vec{b}; z) \right) = \left[z \prod_{i=1}^p (\theta + a_i) - \theta \prod_{i=1}^{p-1} (\theta + b_i - 1) \right] {}_pF_{p-1}(\vec{a}; \vec{b}; z) = 0, \quad (5)$$

where $L(\vec{a}, \vec{b})$ is a differential operator and $\theta = z \frac{d}{dz}$.

Constructing a reduction scheme requires a set of step-up and step-down differential operators for both the upper and lower parameters. In this case, the universal step-up (step-down) operators for the upper (lower) parameters have a very simple form:^b

$$U_{a_i}^+ = \frac{\theta + a_i}{a_i}, \quad L_{b_j}^- = \frac{\theta + b_j - 1}{b_j - 1},$$

and the inverse operators $U_{\vec{a}}^-$, and $L_{\vec{b}}^+$ can be constructed in accordance with Takayama's algorithm. [5] (See also Ref. [2].)

^bSee Ref. [5], or Eq. (2.1), (2.2) in Ref. [2], or Ref. [6] for details.

The differential reduction algorithm takes the form of a product of several differential step-up/step-down operators $U_{\vec{a}}^{\pm}, L_{\vec{b}}^{\pm}$:

$${}_{p+1}F_p(\vec{a} + \vec{m}; \vec{b} + \vec{n}; z) = (U_{\vec{a}}^{\pm})^{\sum_i m_i} (L_{\vec{b}}^{\pm})^{\sum_j n_j} {}_{p+1}F_p(\vec{a}; \vec{b}; z), \quad (6)$$

so that the maximal power of θ in this expression is $r \equiv \sum_i m_i + \sum_j n_j$. Since the hypergeometric function ${}_{p+1}F_p(\vec{a}; \vec{b}; z)$ satisfies a differential equation (5) of order p , it is possible to express all terms containing powers of θ^k with $k \geq p$ in terms of $\theta^j {}_{p+1}F_p(\vec{a}; \vec{b}; z)$ with $j \leq p$, multiplied by coefficients that are rational functions of the parameters and the argument z . In this way, any function ${}_{p+1}F_p(\vec{a} + \vec{m}; \vec{b} + \vec{k}; z)$ may be expressed in terms of a basic function ${}_{p+1}F_p(\vec{a}; \vec{b}; z)$ and its first p derivatives:

$$S(a_i, b_j, z) {}_{p+1}F_p(\vec{a} + \vec{m}; \vec{b} + \vec{k}; z) = \left\{ R_1(a_i, b_j, z)\theta^p + R_2(a_i, b_j, z)\theta^{p-1} + \cdots + R_p(a_i, b_j, z)\theta + R_{p+1}(a_i, b_j, z) \right\} {}_{p+1}F_p(\vec{a}; \vec{b}; z), \quad (7)$$

where \vec{m}, \vec{k} are lists of integers and S and T_i are polynomials in the parameters $\{a_i\}, \{b_j\}$ and z . For some special sets of parameters, the result of the reduction (7) takes a simple form. In particular, when one of the upper parameters is an integer,

$$\tilde{S}(a_i, b_j, z) {}_{p+1}F_p(\vec{j}, \vec{a} + \vec{m}; \vec{b} + \vec{k}; z) = \tilde{R}_1(a_i, b_j, z) + \left\{ \tilde{R}_2(a_i, b_j, z)\theta^{p-1} + \cdots + \tilde{R}_p(a_i, b_j, z)\theta + \tilde{R}_{p+1}(a_i, b_j, z) \right\} {}_{p+1}F_p(\vec{a}; \vec{b}; z).$$

For further details see Ref. [7].

3 Example of reduction

As an example of how the differential reduction applies^c to a particular type of Feynman diagram, let us consider the q -loop bubble diagram B_{112200}^q in Fig. 1, with four massive lines (with masses m, M as indicated) and two sets of massless subloops with r and x lines, respectively. It is defined as

$$\begin{aligned} B_{112200}^q(m^2, M^2, \alpha_1, \alpha_2, \beta_1, \beta_2, \sigma_1, \dots, \sigma_x, \rho_1, \dots, \rho_r) \\ = \int \frac{d^n(k_1 \cdots k_q)}{[k_1^2]^{\rho_1} \cdots [k_{r-1}^2]^{\rho_{r-1}} [(k_1 + \cdots + k_{r-1} + P)^2]^{\rho_r}} \\ \times \frac{1}{[k_{r+x}^2 - m^2]^{\alpha_1} [(k_r + \cdots + k_{r+x} + P)^2 - m^2]^{\alpha_2}} \\ \times \frac{1}{[k_r^2]^{\sigma_1} \cdots [k_{r+x-1}^2]^{\sigma_x} [k_{q-1}^2 - M^2]^{\beta_1} [k_q^2 - M^2]^{\beta_2}}, \end{aligned} \quad (8)$$

^cSee also Ref. [8] for more examples.

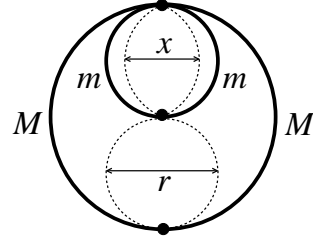


Figure 1: q -loop bubble B_{112200}^q

where $P = k_{q-1} + k_q$ and $x + r = q - 2$, $q \geq 3$. The Mellin-Barnes representation for this diagram is^d

$$\begin{aligned}
& B_{112200}^q(m^2, M^2, \alpha_1, \alpha_2, \beta_1, \beta_2, \sigma_1, \dots, \sigma_x, \rho_1, \dots, \rho_r) \\
&= \frac{[i^{1-n} \pi^{\frac{n}{2}}]^q (-m^2)^{\frac{n}{2}-\alpha_1-\alpha_2} (-M^2)^{\frac{n}{2}(q-1)-\rho-\sigma-\beta_1-\beta_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\frac{n}{2})} \\
&\quad \times \left\{ \prod_{i=1}^x \frac{\Gamma(\frac{n}{2} - \sigma_i)}{\Gamma(\sigma_i)} \right\} \left\{ \prod_{j=1}^r \frac{\Gamma(\frac{n}{2} - \rho_j)}{\Gamma(\rho_j)} \right\} \frac{\Gamma(\rho - \frac{n}{2}(r-1))}{\Gamma(\frac{n}{2}r - \rho)} \\
&\quad \times \int ds \left(\frac{M^2}{m^2} \right)^s \Gamma(\sigma - \frac{n}{2}x - s) \Gamma(\beta_1 + \beta_2 + \rho + \sigma - \frac{n}{2}(q-1) - s) \\
&\quad \times \frac{\Gamma(\alpha_1 + s)\Gamma(\alpha_2 + s)\Gamma(\alpha_1 + \alpha_2 - \frac{n}{2} + s)\Gamma(\frac{n}{2} + s)\Gamma(\frac{n}{2}(q-2) - \rho - \sigma + s)}{\Gamma(\alpha_1 + \alpha_2 + 2s)\Gamma(\frac{n}{2}(x+1) - \sigma + s)} \\
&\quad \times \frac{\Gamma(\beta_1 + \rho + \sigma - \frac{n}{2}(q-2) - s)\Gamma(\beta_2 + \rho + \sigma - \frac{n}{2}(q-2) - s)}{\Gamma(\beta_1 + \beta_2 + 2\sigma + 2\rho - n(q-2) - 2s)}, \tag{9}
\end{aligned}$$

where we have introduced the notations $\rho = \sum_{a=1}^r \rho_a$, $\sigma = \sum_{a=1}^x \sigma_a$.

Closing the contour of integration on the left (see Ref. [9] for details), the result may be expressed as a sum of four hypergeometric functions of type ${}_7F_6$. The general result is too long to reproduce here. In the case that the exponents of the propagators are all integers, the hypergeometric functions appearing in the result are reducible to the following hypergeometric functions and derivatives thereof:^e

$$\begin{aligned}
& \{1, \theta, \theta^2, \theta^3\} \times {}_4F_3 \left(\begin{matrix} I_1 - \frac{n}{2}(x-1), I_2 - \frac{n}{2}x, I_3 - \frac{n}{2}(x+1), \frac{1}{2} + I_4 + \frac{n}{2}(q-x-2) \\ I_5 + \frac{n}{2}, I_6 + \frac{n}{2}(q-x-1), \frac{1}{2} + I_7 - \frac{n}{2}x \end{matrix} \middle| z \right) \\
& \{1, \theta, \theta^2, \theta^3\} \times {}_4F_3 \left(\begin{matrix} \frac{1}{2} + I_1 - \frac{n}{2}, I_2 - \frac{n}{2}(q-2), I_3 - \frac{n}{2}(q-1), I_4 - \frac{n}{2}q \\ I_5 - \frac{n}{2}(q-x-1), I_6 - \frac{n}{2}(q-x-2), \frac{1}{2} + I_7 - \frac{n}{2}(q-1) \end{matrix} \middle| z \right) \\
& \{1, \theta, \theta^2, \theta^3\} \times {}_5F_4 \left(\begin{matrix} 1, \frac{1}{2} + I_1, I_3 - \frac{n}{2}(q-1), I_2 - \frac{n}{2}(q-2), I_4 - \frac{n}{2}(q-3) \\ I_5 + \frac{n}{2}, \frac{1}{2} + I_6 - \frac{n}{2}(q-2), I_7 - \frac{n}{2}(q-x-2), I_8 - \frac{n}{2}(q-x-3) \end{matrix} \middle| z \right), \tag{10}
\end{aligned}$$

where I_k are arbitrary integers. In the last expression, some polynomials are also generated.

Further simplification is possible when q, x take particular values. For example, for $x = 0$ and $x = 1$, the first hypergeometric function is reducible to

$$\{1, \theta\} \times {}_3F_2 \left(\begin{matrix} 1, I_2 - \frac{n}{2}, \frac{1}{2} + I_3 + \frac{n}{2}(q-2) \\ I_4 + \frac{n}{2}(q-1), \frac{1}{2} + I_5 \end{matrix} \middle| z \right) \quad (\text{for } x = 0), \tag{11}$$

or to

$$\{1, \theta, \theta^2\} \times {}_4F_3 \left(\begin{matrix} 1, I_1 - \frac{n}{2}, I_2 - n, \frac{1}{2} + I_3 + \frac{n}{2}(q-3) \\ I_4 + \frac{n}{2}, I_5 + \frac{n}{2}(q-2), \frac{1}{2} + I_6 - \frac{n}{2} \end{matrix} \middle| z \right) \quad (\text{for } x = 1). \tag{12}$$

^dWe note that a smooth limit exists for $\sigma \rightarrow 0$ and $\rho \rightarrow 0$.

^eTwo of the original hypergeometric functions have a similar parameter structure and the same basis functions.

4 Reduction at a singular surface via the ε expansion and hyperlogarithms

In physical applications, the case of equal masses $m^2 = M^2$ (a “single-scale” diagram) is of special interest.^f For the diagram in Fig. (1), this case corresponds to $z = 1$, which is a singular point for the differential reduction algorithm. The question is then how to find a smooth limit at this point. Let us recall [11] that the hypergeometric function (4) converges at $z = 1$ if $\text{Re}(\Sigma b_j - \Sigma a_i) > 0$. In this way, if the hypergeometric function on the l.h.s. of Eq. (7) is well-defined at $z = 1$, a smooth limit of the differential reduction exists. It is now a technical problem to rewrite the r.h.s. of Eq. (7) in terms of the variable $x = 1 - z$. It is well-known, however, that for $p \geq 3$, the hypergeometric function ${}_pF_{p-1}$ is not expressible in terms of hypergeometric functions of the same type in the neighborhood of $z = 1$ (see Ref. [13]).

One approach to this problem is to construct the all-order ε -expansion^g in terms of functions which are defined for the entire range $0 \leq z \leq 1$. The problem is then solved at each order in ε . The hyperlogarithms [16] belong to that class. Unfortunately, at present, the necessary theorems on the all-order ε -expansion are proven only for special sets of parameters [17, 18, 19]. Let us recall some results from Ref. [18]. There are bases (sets of parameters) for which the all-order ε -expansion of a hypergeometric function has the form (see Ref. [18])

$${}_pF_{p-1}(\vec{A} + \varepsilon \vec{a}; \vec{B} + \varepsilon \vec{b}; z) = C(\vec{A}, \vec{a}, \vec{B}, \vec{b}, z) \sum_{j=0}^{\infty} \varepsilon^j \sum_{\vec{J}, k, \vec{s}=1} c_{\vec{s}}(\vec{a}, \vec{b}) \text{Li}_{\vec{s}} \left(\vec{\lambda}_q^{\vec{J}}, \lambda_q^k \xi \right), \quad (13)$$

where $\sum_i s_i = j$, $1 \leq j_a, k \leq q$, and q is integer number, the coefficients $c_{\vec{s}}(\vec{a}, \vec{b})$ are polynomials in the parameters $\{a_j\}$ and $\{b_k\}$, λ_q is primitive q^{th} -root of unity, $\text{Li}_{\vec{s}} \left(\vec{\lambda}_q^{\vec{J}}, \lambda_q^k \xi \right)$ are hyperlogarithms, $\vec{\lambda}_q^{\vec{J}}$ is short-hand for $\lambda_q^{j_1 - j_2}, \lambda_q^{j_2 - j_3}, \dots, \lambda_q^{j_{k-1} - j_k}$, ξ is a variable related algebraically^h to z , and $C(\vec{A}, \vec{a}, \vec{B}, \vec{b}, z)$ is a polynomial. Thus, at each order in the ε -expansion, only hyperlogarithms of a single weight are generated. The definition of the hyperlogarithm as an iterated integral over any rational function

$$\begin{aligned} I(z; a_k, a_{k-1}, \dots, a_1) &= \int_0^z \frac{dt_k}{t_k - a_k} \int_0^{t_k} \frac{dt_{k-1}}{t_{k-1} - a_{k-1}} \dots \int_0^{t_2} \frac{dt_1}{t_1 - a_1} \\ &= \int_0^z \frac{dt}{t - a_k} I(t; a_{k-1}, \dots, a_1), \end{aligned} \quad (14)$$

where the a_j are arbitrary numbers, together with the structure of the expansion (13), show that the transformation $z \rightarrow 1 - z$ results in functions of the same structure. In this way, we can construct the necessary limiting procedure without detailed knowledge about

^f See Ref. [10] for the three-loop case ($q = 3, x = 0, r = 0$) and Ref. [12] for the four-loop case ($q = 4, x = 1, r = 0$).

^g See Ref. [14] for another technique of evaluating hypergeometric functions at $z = 1$.

^h For completeness, we note that ξ can take the explicit forms $\xi_{1,2,3} = z^{\frac{1}{q}}, (1-z)^{\frac{1}{q}}, \left(\frac{z}{z-1}\right)^{\frac{1}{q}}$ where q is an integer. Under $z \rightarrow 1 - z$, these variables transform as $\xi_{1,2} \rightarrow \xi_{2,1}$ and $\xi_3 \rightarrow \frac{1}{\xi_3}$.

the relationship between the functions ${}_pF_{p-1}(\vec{a}; \vec{b}; z)$ and ${}_pF_{p-1}(\vec{a}; \vec{b}; 1 - z)$. For practical applications to diagrams presently of interest, a few coefficients of the ε -expansion suffice, and these have been implemented in several existing packages [20].

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